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The Smallest Rounded Sets of Binary Matroids

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It was proved by Oxley that $U_{2,4}$ is the only non-trivial 3-connected matroid N such that, whenever a 3-connected matroid M has an N-minor and x and y are elements of M, there is an N-minor of M using $\{x, y\}$. This paper establishes the corresponding result for binary matroids by proving that if M and N above must both be binary, then there are exactly two possibilities for N: the rank-three and rank-four wheels.

1. INTRODUCTION

The property of roundedness in matroids is concerned with relating the existence of certain minors in a matroid to particular elements of the matroid. This property has been studied by a number of authors [1–3, 6, 8, 10–19] and its role in the study of matroid structure was surveyed by Seymour [20, Section 3].

Most of the matroid terminology used here follows Welsh [22]. If X is a subset of the ground set $E(M)$ of a matroid M, we shall say that M *uses* X. The deletion and contraction of X from M will be denoted by $M\backslash X$ and M/X, respectively. The closure and rank of X in M will be denoted by $\sigma_M X$ and $\mathrm{rk}_M X$. We shall write $\mathrm{rk}\, M$ for $\mathrm{rk}_M E(M)$. A three-element circuit of M will be called a *triangle* and a three-element cocircuit a *triad*. The property that M has no circuit and cocircuit with exactly one common element will be referred to as *orthogonality*.

If M_1 and M_2 are matroids on the sets S and $S \cup e$ where $e \notin S$, then M_2 is called an *extension* of M_1 if $M_2\backslash e = M_1$, and M_2 is called a *lift* of M_1 if $M_2/e = M_1$. We call M_2 a *non-trivial extension* of M_1 if e is neither a loop nor a coloop of M_2 and e is not in a 2-element circuit of M_2.

Let $\mathcal{S}$ be a set of matroids. a matroid M' is an $\mathcal{S}$*-minor* of M if M' is a minor of M isomorphic to some member of $\mathcal{S}$. Let k and m be positive integers. Then $\mathcal{S}$ is (k, m)*-rounded* [3] if every member of $\mathcal{S}$ is k-connected [22, p. 79] having at least four elements and the following condition holds:

(1.1) *If M is a k-connected matroid having an $\mathcal{S}$-minor and X is a subset of $E(M)$ with at most m elements, then M has an $\mathcal{S}$-minor using X.*

Bixby [2] and Seymour [17], respectively, proved that $\{U_{2,4}\}$ is $(2,1)$- and $(3,2)$-rounded, while Oxley [10, (1.5)] extended Seymour's result by proving the following:

(1.2) THEOREM. *$\{N\}$ is $(3, 2)$-rounded iff N is isomorphic to $U_{2,4}$.*

The next theorem, the main result of this paper, determines all 2-element sets $\mathcal{S}$ that are $(3,2)$-rounded. We denote by $\mathcal{W}_r$ the r-spoked wheel graph and by $\mathcal{W}^r$ the rank-r whirl [22, pp. 80–81]. Note that $\mathcal{W}^2 \simeq U_{2,4}$.

(1.3) THEOREM. *Let M and N be non-isomorphic matroids. Then $\{M, N\}$ is*

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(3, 2)-rounded iff $\{M, N\}$ *is* $\{U_{2,4}, M'\}$ *where either:*
(i) M' *is non-binary and 3-connected; or*
(ii) M' *is isomorphic to* $M(\mathcal{W}_3)$ *or* $M(\mathcal{W}_4)$.

A set $\mathcal{S}$ of k-connected matroids each having at least four elements will be called *(k, m)-rounded within the class of GF(q)-representable matroids* if every member of $\mathcal{S}$ is $GF(q)$-representable, and (1.1) holds for all $GF(q)$-representable matroids M. A key step in the proof of Theorem 1.3 involves establishing that the only singleton sets that are (3, 2)-rounded within the class of binary matroids are $\{M(\mathcal{W}_3)\}$ and $\{M(\mathcal{W}_4)\}$. This is an immediate consequence of the following result.

(1.4) THEOREM. $\{N\}$ *is* (3, 2)*-rounded within the class of* $GF(q)$*-representable matroids iff either:*
(i) $q = 2$ *and* N *is isomorphic to* $M(\mathcal{W}_3)$ *or* $M(\mathcal{W}_4)$*; or*
(ii) $q = 3$ *and* N *is isomorphic to* $U_{2,4}$ *or* $\mathcal{W}^3$*; or*
(iii) $q \notin \{2, 3\}$ *and* N *is isomorphic to* $U_{2,4}$.

The proofs of Theorems 1.3 and 1.4 will be given in Sections 3 and 2, respectively. These proofs will use Crapo's theory of modular cuts (see, for example, [22, p. 320]), the basis of which is that an extension M_1 of a matroid M by an element e is uniquely determined by the set $\mathcal{M}$ of flats F of M such that the flat $F \cup e$ of M_1 has the same rank as F. The set $\mathcal{M}$ here is called a *modular cut,* and Crapo [7] determined precisely which sets of flats can form modular cuts. It follows from this result that the intersection of two modular cuts is also a modular cut. If $\mathcal{F}$ is a set of flats of M, the modular cut *generated* by $\mathcal{F}$ is the intersection of all modular cuts containing $\mathcal{F}$. The extension M_1 of M corresponding to the modular cut $\mathcal{M}$ will be said to be *determined* by $\mathcal{M}$ and, if $E(M_1) - E(M) = \{e\}$, we shall write $M + e$ for M_1. If $\mathcal{M} = \{E(M)\}$, then we say that e is *freely added* to M or is *free* in M_1. Evidently e is freely added to M iff $\operatorname{rk} M_1 = \operatorname{rk} M$ and every circuit of M_1 containing e has size $\operatorname{rk} M + 1$.

The remainder of this section will state various results that will be used in the proofs of the main theorems. The first such result is Seymour's quick test for (3, 2)-roundedness.

(1.5) THEOREM [18]. *Let* $\mathcal{S}$ *be a set of* 3*-connected matroids each having at least four elements. Then* $\mathcal{S}$ *is* (3, 2)*-rounded iff the following condition holds: if* M *is a* 3*-connected extension or lift of a member of* $\mathcal{S}$ *and* X *is a* 2*-element subset of* M*, then* M *has an* $\mathcal{S}$*-minor using* X.

Throughout the proof of Theorem 1.3, the following result will be frequently used in the construction of extensions.

(1.6) LEMMA [10, (2.5)]. *Let* $\{x_1, x_2, \ldots, x_n\}$ *be a circuit in a matroid* M *and suppose that* x_1 *is in every dependent flat of* M*. Then a flat* F *of* M *is in the modular cut* $\mathcal{M}$ *generated by* $\sigma_M\{x_1, x_2\}$ *and* $\sigma_M\{x_3, x_4, \ldots, x_n\}$ *iff* F *contains one of the two generating flats. Moreover, the generating flats are disjoint.*

We observe that, in the last lemma, if $n \geq 4$ and M is 3-connected, then the extension M_1 of M determined by $\mathcal{M}$ is non-trivial. Hence M_1 is 3-connected (see, for example, [9, Lemma 2.1]).

Duality will be frequently invoked in the proofs of the main result. In particular, we shall use the elementary fact that a set $\{M_1, M_2, \ldots, M_j\}$ of matroids is (k, m)-

rounded iff the set $\{M_1^*, M_2^*, \ldots, M_j^*\}$ is (k, m)-rounded. We shall also use the next two results that relate free elements to duality. The first of these follows easily from Lemma 2.2 of [10]. The elementary proof of the second is omitted.

(1.7) LEMMA. *Let e be an element of a connected matroid M having at least two elements. Then e is free in M^* iff e is in every dependent flat of M.*

In view of this result, if M is connected and $|E(M)| \geq 2$, an element that is in every dependent flat of M will be called a *cofree* element of M.

(1.8) LEMMA. *Let M be a connected matroid with at least two elements. Then M has an element that is both free and cofree if and only if $M \simeq U_{r,n}$ for some integer r such that $1 \leq r \leq n-1$.*

2. ROUNDEDNESS IN $GF(q)$-REPRESENTABLE MATROIDS

In this section we prove Theorem 1.4. We shall use the following result.

(2.1) LEMMA. *Let N be a whirl or the cycle matroid of a wheel. Then $\{N\}$ is $(3, 2)$-rounded within the class of $GF(q)$-representable matroids iff one of (i)–(iii) of (1.4) holds.*

PROOF. Seymour [18] and Reid [15], respectively, showed that $\{U_{2,4}, M(\mathscr{W}_r)\}$ is $(3,2)$-rounded when r is 3 and when r is 4. It follows immediately from this that both $\{M(\mathscr{W}_3)\}$ and $\{M(\mathscr{W}_4)\}$ are $(3,2)$-rounded within the class of binary matroids. It is straightforward to show that, when $r \geq 5$ and $q \geq 2$ and when $r \in \{3, 4\}$ and $q \geq 3$, $\{M(\mathscr{W}_r)\}$ is not $(3,2)$-rounded within the class of $GF(q)$-representable matroids. We omit the details.

Now consider the whirls recalling that $\mathscr{W}^2 \simeq U_{2,4}$. Since $\{U_{2,4}\}$ is $(3,2)$-rounded, it is $(3,2)$-rounded within the class of $GF(q)$-representable matroids provided that $U_{2,4}$ is $GF(q)$-representable; that is, provided that $q \geq 3$. Next consider $\{\mathscr{W}^r\}$ for $r \geq 3$. It follows from [11, Lemma 3.4] that, when r is 3, this set is $(3,2)$-rounded within the class of ternary matroids. Moreover, it is not difficult to show that when $r = 3$ and $q \geq 4$ and when $r \geq 4$ and $q \geq 2$, $\{\mathscr{W}^r\}$ is not $(3,2)$-rounded within the class of $GF(q)$-representable matroids. Again we omit the details. □

Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, k-1\}$,

(2.2.1) exactly one of T_i and T_{i+1} is a triangle;

(2.2.2) $|T_i \cap T_{i+1}| = 2$; and

(2.2.3) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \cdots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of M of *length* k. Evidently $T_1, T_2, \ldots, T_k$ is a chain of M iff it is a chain of M^*.

PROOF OF THEOREM 1.4. Let $\operatorname{rk} N = r$ and let $V(r, q)$ denote the r-dimensional vector space over $GF(q)$. Evidently we may identify N with the restriction of the matroid $V(r, q)$ to some set S.

We shall first show that N has a triangle. Let $\{c_1, c_2, \ldots, c_j\}$ be a circuit of N of minimum size and suppose that $j \geq 4$. Let L be the line of $V(r, q)$ that is spanned by

$\{c_1, c_2\}$ and let N' be the restriction $V(r, q) \mid (S \cup L)$. Now L is a modular flat of $V(r, q)$ and is therefore a modular flat of N' [5]. It follows that L meets $\sigma_{N'}\{c_3, c_4, \ldots, c_j\}$. Thus, for some v in $L - \{c_1, c_2\}$, both $\{c_1, c_2, v\}$ and $\{v, c_3, c_4, \ldots, c_j\}$ are circuits of N', and so both these sets are circuits of N'' where N'' is $V(r, q) \mid (S \cup v)$. Hence any single-element deletion of N'' that uses v has a circuit of size less than j. Thus the 3-connected matroid N'' has no N-minor using v—a contradiction. We conclude that N has a triangle. Hence N has a chain.

Let $T_1, T_2, \ldots, T_k$ be a chain of N of maximum length. A straightforward induction argument using orthogonality gives that $T_1 \cup T_2 \cup \cdots \cup T_k$ has $k+2$ distinct elements $a_1, a_2, \ldots, a_{k+2}$ such that, for all i in $\{1, 2, \ldots, k\}$, $T_i = \{a_i, a_{i+1}, a_{i+2}\}$. By duality, we may assume that T_k is a triad of N.

Now let L be the line of $V(r, q)$ spanned by $\{a_{k+1}, a_{k+2}\}$ and let $L - \{a_{k+1}, a_{k+2}\}$ be $\{v_1, v_2, \ldots, v_{q-1}\}$. We shall show that $\{v_1, v_2, \ldots, v_{q-1}\}$ meets the ground set S of N. Assume the contrary and let N_i be $V(r, q) \mid (S \cup v_i)$. Then either:

(I) for some i in $\{1, 2, \ldots, q-1\}$, T_k is a cocircuit of N_i; or
(II) for all such i, $T_k \cup v_i$ is a cocircuit of N_i.

Assume that (I) holds. Evidently $\{a_{k+1}, a_{k+2}, v_i\}$ is a triangle of N_i. By using orthogonality and the fact that T_k is a triad of N_i, we deduce that if T_j is a triad of N, it is a triad of N_i. Evidently N_i has an N-minor using both a_1 and v_i. Thus there is an element g of $E(N_i) - \{a_1, v_i\}$ such that $N_i \backslash g \simeq N$. Since every element of $\{a_2, a_3, \ldots, a_{k+2}\}$ is in a triad of N_i and $N_i \backslash g$ is 3-connected, $g \notin \{a_2, a_3, \ldots, a_{k+2}\}$. Hence $T_1, T_2, \ldots, T_k, \{a_{k+1}, a_{k+2}, v_i\}$ is a chain of $N_i \backslash g$ of length $k+1$. As $N_i \backslash g \simeq N$, the latter has a chain of length $k+1$—a contradiction.

Now suppose that (II) holds. Let $N' = V(r, q) \mid (S \cup L)$. Since L is a modular line of N', it meets every hyperplane of N'. Therefore L is not contained in any cocircuit of N'. Hence $L \cup a_k$ is not a cocircuit of N'. Since $T_k \cup v_1$ is a cocircuit of N_1 and $N_1 = N' \backslash \{v_1, v_3, \ldots, v_{q-1}\}$, it follows that N' has a cocircuit C_1^* that contains $T_k \cup v_1$ and is contained in $L \cup a_k$. Since $C_1^* \neq L \cup a_k$, we may assume, without loss of generality, that $v_2 \notin C_1^*$. But N' has a cocircuit C_2^* that contains $T_k \cup v_2$ and is contained in $L \cup a_k$. Therefore, by cocircuit elimination, N' has a cocircuit C^* that is contained in L. Since every 3-element subset of L is a circuit of N', orthogonality implies that C^* avoids at most one element of L. In particular, C^* contains a_{k+1} or a_{k+2}, and so $\{a_{k+1}, a_{k+2}\}$ contains a cocircuit of N. This contradiction completes the proof that neither (I) nor (II) holds. We conclude that $\{v_1, v_2, \ldots, v_{q-1}\} \cap S$ must be non-empty. Assume, without loss of generality, that $v_1 \in S$.

Since $\{a_{k+1}, a_{k+2}, v_1\}$ is a triangle of N and $T_1, T_2, \ldots, T_k$ is a maximum-length chain, $v_1 \in T_1 \cup T_2 \cup \cdots \cup T_k$. Every element of $(T_1 \cup T_2 \cup \cdots \cup T_{k-2}) - \{a_1\}$ is in a triad of N that does not contain a_{k+1} or a_{k+2}. Thus, by orthogonality, $v_1 \notin (T_1 \cup T_2 \cup \cdots \cup T_{k-2}) - \{a_1\} = \{a_2, a_3, \ldots, a_k\}$. Since v_1 is clearly not a_{k+1} or a_{k+2}, we conclude, as $v_1 \in \{a_1, a_2, \ldots, a_{k+2}\}$, that $v_1 = a_1$. Moreover, T_1 is a triangle of N and k is even.

Now let T_{k+1} be the triangle $\{a_{k+1}, a_{k+2}, a_1\}$ of N. Then $T_2, T_3, \ldots, T_{k+1}$ is a maximum-length chain of N. Hence $T_2, T_3, \ldots, T_{k+1}$ is a maximum-length chain of N^* having T_{k+1} as a triad. Applying the above argument to this chain, we deduce that $\{a_{k+2}, a_1, a_2\}$ is a triangle of N^*; that is, this set is a triad of N.

Now let $A = \{a_1, a_2, \ldots, a_{k+2}\}$. Then A is spanned in N and N^* by $\{a_1, a_3, a_5, \ldots, a_{k+1}\}$ and $\{a_2, a_4, a_6, \ldots, a_{k+2}\}$, respectively. Thus

$$\mathrm{rk}_N A + \mathrm{rk}_{N^*} A - |A| \leq 0.$$

Rewriting the left-hand side here, we have

$$\mathrm{rk}_N A + \mathrm{rk}_N(E(N) - A) - \mathrm{rk}\, N \leqslant 0.$$

Therefore, as N is connected, we conclude that $A = E(N)$. Hence every element of the 3-connected matroid N is in both a triangle and a triad. Thus $N \simeq U_{2,4}$, or, by Tutte's wheels and whirls theorem [21], $\mathrm{rk}\, N \geqslant 3$ and N is isomorphic to a whirl or the cycle matroid of a wheel. The theorem now follows by Lemma 2.1. □

3. The Proof of Theorem 1.3

By Lemma 2.1 and Theorem 1.2, if $\{M, N\} = \{U_{2,4}, M'\}$, where M' is isomorphic to $M(\mathscr{W}_3)$, $M(\mathscr{W}_4)$ or some non-binary 3-connected matroid, then $\{M, N\}$ is $(3, 2)$-rounded. The remainder of this section is devoted to proving the converse of this. Throughout, M and N will denote non-isomorphic matroids for which $\{M, N\}$ is $(3, 2)$-rounded. If $M \simeq U_{2,4}$, then we may assume that N is binary. But then, by Theorem 1.4, N is isomorphic to $M(\mathscr{W}_3)$ or $M(\mathscr{W}_4)$. It follows that we may suppose that neither M nor N is isomorphic to $U_{2,4}$.

(3.1) Lemma. *Both M and N have rank and corank at least* 3.

Proof. By duality, it suffices to show that neither M nor N has rank 2. We shall prove a stronger result. For $n \geqslant 5$, let Q_{n+1} be the rank-3 matroid for which a Euclidean representation is shown in Figure 1.

(3.2) Lemma. *If $n \geqslant 5$, then neither M nor N is isomorphic to $U_{2,n}$ or Q_{n+1}.*

Proof. Assume the contrary and let

$$m = \min\{n\colon M \text{ or } N \text{ is isomorphic to } U_{2,n} \text{ or } Q_{n+1}\}.$$

Evidently $m \geqslant 5$. Suppose that $M \simeq U_{2,m}$. Then Q_{m+1} has an M-minor but has no such minor using $\{e, f\}$. Hence Q_{m+1} has an N-minor using $\{e, f\}$. By the choice of m, it follows that $N \simeq Q_{m+1}$. But now the matroid D_{m+2} in Figure 2 has an N-minor, yet has

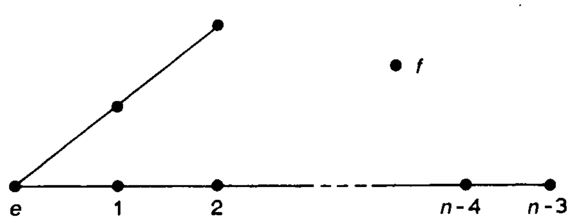

Figure 1.

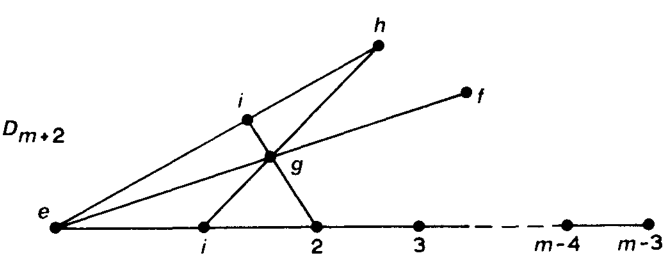

Figure 2.

no M- or N-minor using $\{e, g\}$. This contradiction implies that $M \not\cong U_{2,m}$. Similarly, $N \not\cong U_{2,m}$.

We may now assume that $M \cong Q_{m+1}$. It follows that D_{m+2} has an N-minor using $\{e, g\}$. By the choice of m, $\operatorname{rk} N = 3$. Thus D_{m+2} has a restriction N_1 that uses $\{e, g\}$ and is isomorphic to N. Since N_1 has no 2-element cocircuits, $E(N_1)$ uses at least two of i, h and f. It follows, since N_1 is 3-connected, that it has at most one free element.

Next consider the matroid $Q_{m+1} + j$ that is obtained from Q_{m+1} by freely adding j. This matroid has no Q_{m+1}-minor using $\{f, j\}$ and so must have a restriction isomorphic to N using $\{f, j\}$. In such a restriction, f and j are free. Hence N_1 has at least two free elements. This contradiction completes the proof of Lemma 3.2 and thereby that of Lemma 3.1. □

The next two results are steps towards Lemma 3.5, which shows that M and N have the same number of elements. Although the following lemma is not explicitly stated in [10], it is not difficult to see that it may be obtained from the proof of Lemma 2.6 of that paper. We note that Q_7^* may be constructed from the parallel connection [4] of a triangle and a 4-element circuit by freely adding an element.

(3.3) LEMMA. *Let N_1 be a 3-connected matroid having rank and corank at least three and assume that N_1 has both a free element and a cofree element. Suppose that, whenever N_2 is a non-trivial extension of N_1, each element of N_2 appears in an N_1-minor. Then N_1 is isomorphic to Q_6 or Q_7^*.*

(3.4) LEMMA. *(i) M or N has at least two free elements*; *and (ii) neither M nor N is a lift or an extension of the other.*

PROOF. Part (i) follows easily from considering the matroid obtained from M by freely adding two elements. To prove (ii), suppose that $M/e \cong N$ and let $N + f$ be formed by freely adding f to N. Evidently $N + f$ has an N-minor using f, so N has a free element. As $\{M^*, N^*\}$ is $(3, 2)$-rounded, we may apply (i) to it to obtain that M^* or N^* has at least two free elements. Since $N^* \cong M^* \backslash e$, it follows, in either case, that N^* has a free element. Thus, by Lemma 3.3, $N \cong Q_6$ or Q_7^*. But, by Lemma 3.2 and duality, this is a contradiction. We conclude that M is not a lift of N and, by duality, M is not an extension of N. □

(3.5) LEMMA. $|E(M)| = |E(N)|$.

PROOF. By Lemma 3.1 and Theorem 1.2 neither $\{M\}$ nor $\{N\}$ is $(3, 2)$-rounded. Thus, if $|E(N)| < |E(M)|$, then, by Theorem 1.5, M is an extension or lift of N. But this contradicts Lemma 3.4(ii). It follows that $|E(N)| \geqslant |E(M)|$ and, likewise, $|E(M)| \geqslant |E(N)|$. □

The next step in the proof of Theorem 1.3 is to show that M and N have the same rank. To prove this, we shall need the following lemma that will also be used in the proof of Theorem 3.9.

(3.6) LEMMA. *M, N, M^* or N^* has at least one free element and at least two cofree elements.*

PROOF. By Lemma 3.4(i) and duality, at least one member of each of $\{M, N\}$ and $\{M^*, N^*\}$ has two or more free elements. Thus either the lemma holds or we may assume, without loss of generality, that both M and N^* have at least two free elements.

Let $N+f$ be formed by freely adding f to N. If $N+f$ has an N-minor using f, then N has the required property. Thus we may assume that $N+f$ has no such minor. Then $N+f$ has an N-minor using f. Since $|E(M)| = |E(N)|$, this M-minor has at least one cofree element. Thus M^* has the required property. □

(3.7) LEMMA. $\operatorname{rk} M = \operatorname{rk} N$.

PROOF. Assume, without loss of generality, that $\operatorname{rk} N < \operatorname{rk} M$. Then $\operatorname{rk} M^* < \operatorname{rk} N^*$. By Lemma 3.6, N or M^* has both a free element and a cofree element. Since $|E(M)| = |E(N)|$, it follows, by Theorem 1.2 and Lemma 3.3, that N or M^* is isomorphic to Q_6 or Q_7^*. By Lemma 3.2 and duality, this is a contradiction. □

We shall require one more lemma to prove the main result of this section, namely that $\{M, N\}$ is not $(3, 2)$-rounded.

(3.8) LEMMA. *Both the rank and corank of M and N exceed three.*

PROOF. Assume that the lemma is false. Then, by duality and Lemmas 3.1, 3.5 and 3.7, we may assume that $\operatorname{rk} M = \operatorname{rk} N = 3$, and M and N have the same number, say n, of elements. By Lemmas 3.4(i) and 1.7 and duality, M or N, say N, has at least two elements that are in every dependent flat. Then N has at most one dependent line. Thus, as N has rank 3 and corank at least 3, either $N \simeq U_{3,n}$ for some $n \geq 6$, or, for some i in $\{3, 4, \ldots, n-3\}$, N is isomorphic to the rank-3 matroid L_i that consists of an i-point line and $n-i$ free elements.

Suppose that $N \simeq U_{3,n}$ and let Z be the rank-3 $(n+1)$-point matroid shown in Figure 3. As Z has an N-minor but has no N-minor using e, the matroid Z has an M-minor using e. Therefore, since $\operatorname{rk} M = 3$ and $|E(M)| = n$, M is isomorphic to one of the two non-isomorphic single-element deletions of Z that use $\{e\}$. It is not difficult to check that, in each such case, $\{M, N\}$ is not $(3, 2)$-rounded. Thus $N \not\simeq U_{3,n}$. A similar argument shows that $N \not\simeq L_i$ for any i in $\{3, 4, \ldots, n-3\}$. □

(3.9) THEOREM. *$\{M, N\}$ is not $(3, 2)$-rounded.*

PROOF. By duality and Lemmas 1.7 and 3.6, we may assume that

(3.10) *M has a free element f together with elements d_1 and d_2 which are in every dependent flat.*

We remark that, throughout this proof, condition (3.10) will provide the sole feature distinguishing M from N. Furthermore, we note that, by Lemma 1.8, we may suppose that f, d_1 and d_2 are distinct.

As $\operatorname{rk} M \neq 2$, $f \notin \sigma_M\{d_1, d_2\}$. Now augment $\{d_1, d_2\}$ to a base $\{d_1, d_2, a_1, a_2, \ldots, a_{r-2}\}$ of $M \backslash f$. Let $\mathcal{M}$ be the modular cut of M generated by the flats $\sigma_M\{d_1, d_2\}$

Z

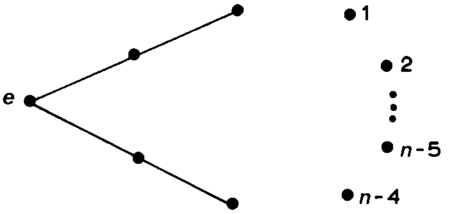

FIGURE 3.

and $\{a_1, a_2, \ldots, a_{r-2}, f\}$ and let $M + e_1$ be the extension determined by $\mathcal{M}$. Evidently $M + e_1$ is 3-connected. Moreover, by Lemma 1.6, we have:

(3.11) *The dependent flats of $M + e_1$ are the circuit-hyperplane $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ together with all the sets $F \cup e_1$ for which F is a flat of M containing $\{d_1, d_2\}$.*

As $\{M, N\}$ is $(3, 2)$-rounded, there is an element g_1 of $E(M + e_1) - \{e_1, f\}$ such that $(M + e_1)\backslash g_1$ is isomorphic to M or N. We now eliminate the first possibility. Thus assume that $(M + e_1)\backslash g_1 \cong M$. We shall show that this implies the contradiction that $(M + e_1)\backslash g_1$ has more dependent flats than M. First note that, as d_1 and d_2 are in every dependent flat of M, no line of M has more elements than $\sigma_M\{d_1, d_2\}$. Thus $g_1 \in \sigma_M\{d_1, d_2\}$. Using this, it is not difficult to check that, for every dependent flat F of M, $(F - g_1) \cup e_1$ is a dependent flat of $(M + e_1)\backslash g_1$. Moreover, $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ is also a dependent flat of $(M + e_1)\backslash g_1$ since $g_1 \notin \{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$. Thus $(M + e_1)\backslash g_1$ does indeed have more dependent flats than M. We conclude that

(3.12) $(M + e_1)\backslash g_1 \cong N$.

As e_1 is in every dependent flat of $(M + e_1)\backslash g_1$, it follows by (3.12) that

(3.13) *N has an element that is in every dependent flat.*

We show next that

(3.14) LEMMA. *N has a unique dependent line.*

PROOF. We shall first show that M or N has a triangle. Among all the circuits of M and N, let $\{c_1, c_2, \ldots, c_j\}$ be one of minimum size and suppose that $j \geq 4$. Let P be the member of $\{M, N\}$ that contains $\{c_1, c_2, \ldots, c_j\}$. As both M and N have an element in every dependent flat, we may assume that c_1 is in every dependent flat of P.

Let $\mathcal{P}$ be the modular cut of P generated by $\sigma_P\{c_1, c_2\}$ and $\sigma_P\{c_3, c_4, \ldots, c_j\}$, and let $P + e_2$ be the extension determined by $\mathcal{P}$. By Lemma 1.6, both $\{c_1, c_2, e_2\}$ and $\{c_3, c_4, \ldots, c_j, e_2\}$ are circuits of $P + e_2$. Hence $P + e_2$ has no M- or N-minor using e_2—a contradiction. We conclude that M or N has a triangle.

Now, as d_1 and d_2 are in every dependent flat of M, by (3.11), the only possible dependent line of $(M + e_1)\backslash g_1$ is $(\sigma_M\{d_1, d_2\} \cup \{e_1\}) - \{g_1\}$. Since M or N has a triangle and $(M + e_1)\backslash g_1 \cong N$, we deduce that $(M + e_1)\backslash g_1$, and hence N, has exactly one dependent line. □

(3.15) LEMMA. $g_1 \in \{a_1, a_2, \ldots, a_{r-2}\}$.

PROOF. Assume the contrary and let $N' = (M + e_1)\backslash g_1$. Then N' has $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ as a circuit-hyperplane. Since $N' \cong N$, the former has a unique dependent line L. By (3.11) and Lemma 3.8, it follows that $L = (\sigma_M\{d_1, d_2\} \cup \{e_1\}) - \{g_1\}$. Moreover, e_1 is in every dependent flat of N'.

Now let $N' + e_3$ be the extension determined by the modular cut generated by the flats $\{e_1, f\}$ and $\{a_1, a_2, \ldots, a_{r-2}\}$ of N'. By Lemma 1.6, $\{e_1, f, e_3\}$, $\{a_1, a_2, \ldots, a_{r-2}, e_3\}$ and L are dependent flats of $N' + e_3$. Moreover, $\{e_1, f, e_3\} \cap L = \{e_1\}$ and $\{a_1, a_2, \ldots, a_{r-2}, e_3\} \cap L$ is empty. As $\{M, N\}$ is $(3, 2)$-rounded, there is an element g_3 of $E(N' + e_3) - \{e_3, e_1\}$ such that $(N' + e_3)\backslash g_3$ is isomorphic to M or N. Since $(N' + e_3)\backslash g_3$ clearly does not have two elements in every dependent flat, (3.10) implies that $(N' + e_3)\backslash g_3 \not\cong M$.

We may now assume that $(N' + e_3)\backslash g_3 \cong N$. By Lemma 3.14, $g_3 \in L \cup \{e_1, f, e_3\}$. But $g_3 \notin \{e_1, e_3\}$ and, by (3.13), $g_3 \neq f$. Hence $g_3 \in L - e_1$. Thus $\{a_1, a_2, \ldots, a_{r-2}, e_3\}$ is both a circuit and a flat of $(N' + e_3)\backslash g_3$. But $(N' + e_3)\backslash g_3 \cong N \cong (M + e_1)\backslash g_1 = N'$ and

$(\sigma_M\{d_1, d_2\} \cup \{e_1\}) - \{g_1\}$ is a dependent line of N'. Thus, by (3.11), the only circuit-flats that $(M+e_1)\backslash g_1$ can contain are a triangle and a hyperplane. Since $\{a_1, a_2, \ldots, a_{r-2}, e_3\}$ has $\operatorname{rk} N - 1$ elements, this set is not a circuit-hyperplane. It must therefore be a triangle, so $r = 4$ and both $\{a_1, a_2, e_3\}$ and $\{e_1, f, e_3\}$ are lines of $(N' + e_3)\backslash g_3$. Since this matroid is isomorphic to N, this contradicts the fact that N has a unique dependent line. □

By (3.11), the only circuit of $M + e_1$ containing f and having fewer than $r+1$ elements is $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$. As $g_1 \in \{a_1, a_2, \ldots, a_{r-2}\}$, it follows that f is free in $(M+e_1)\backslash g_1$ and also, by (3.11), that $(M+e_1)\backslash g_1$ has at least two elements which are in every dependent flat. Since $N \cong (M+e_1)\backslash g_1$, we deduce that N satisfies condition (3.10). Thus M and N obey the same hypotheses. Therefore we may interchange them from (3.10) onward to deduce from Lemma 3.14 that M has a unique dependent line L_M. Evidently $L_M = \sigma_M\{d_1, d_2\}$. As $g_1 \in \{a_1, a_2, \ldots, a_{r-2}\}$, the set $\sigma_M\{d_1, d_2\} \cup \{e_1\}$ is a dependent line of $(M+e_1)\backslash g_1$. Since the last matroid is isomorphic to N, which has a unique dependent line L_N, we deduce that $|L_N| > |L_M|$. But again, since M and N obey the same hypotheses, we may interchange them from (3.10) onward to get that $|L_M| > |L_N|$. This contradiction completes the proof of Theorem 3.9 as well as that of Theorem 1.3. □

4. Some Consequences

In this section we briefly note some consequences of our main results. The first of these is a result of Reid [14], the converse of which was proved in [11].

(4.1) Corollary. *Let M_1 and M_2 be matroids for which $\{M_1, M_2\}$ is (3, 3)-rounded. Then $\{M_1, M_2\}$ is $\{U_{2,4}, \mathcal{W}^3\}$.*

Proof. As $\{M_1, M_2\}$ is (3, 3)-rounded, it is certainly (3, 2)-rounded. Thus, by Theorems 1.2 and 1.3, we may assume that $M_1 \cong U_{2,4}$. Now one easily checks that the only 3-connected monors of $\mathcal{W}^3$ with four or more elements are isomorphic to $U_{2,4}$ and $\mathcal{W}^3$. Moreover, it is not difficult to find a 3-element subset of $E(\mathcal{W}^3)$ that is in no $U_{2,4}$-minor (see, for example, [11]). □

Reid actually proved a slightly more general result than (4.1), since he allowed the members of a (k, m)-rounded set to have fewer than four elements. If we also allow this here, it is straightforward to show that the only additional (3, 2)-rounded pairs obtained are $\{U_{1,2}, U_{0,1}\}$, $\{U_{1,2}, U_{1,1}\}$ and $\{U_{1,3}, U_{2,3}\}$ together with all the pairs $\{M, N\}$ for which M is in $\{U_{1,2}, U_{1,3}, U_{2,3}\}$ and N is an arbitrary 3-connected matroid with at least four elements.

One can use Theorem 1.4 to characterize all singleton sets that are (3, 3)-rounded within the class of $GF(q)$-representable matroids:

(4.2) Corollary. *$\{N\}$ is (3, 3)-rounded within the class of $GF(q)$-representable matroids iff $q = 2$ and $N \cong M(\mathcal{W}_3)$.*

Proof. By [11, Theorem 3.6], $\{M(\mathcal{W}_3)\}$ is (3, 3)-rounded within the class of binary matroids. It is straightforward to check that none of the other possibilities listed in Theorem 1.4 is (3, 3)-rounded within the class of $GF(q)$-representable matroids for the specified values of q. □

Finally, we remark that it is easy to modify the proof of Theorem 1.4 to obtain the following roundedness result for the class of graphic matroids.

(4.3) THEOREM. *Let H be a 3-connected simple graph having at least four vertices. Suppose that, whenever G is a 3-connected simple graph having an H-minor and $\{x, y\} \subseteq E(G)$, there is an H-minor of G using $\{x, y\}$. Then H is isomorphic to $\mathcal{W}_3$ or $\mathcal{W}_4$.*

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